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Solutions to Semilinear Elliptic Problems with Combined Nonlinearities[1]

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We are concerned with the following nonlinear Dirichlet problem:

$$\begin{cases} -\Delta u = h(x)u^q + f(x,u), \\ 0 \leqslant u \in H_0^1(\Omega), \qquad 0 < q < 1, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ $(N \geqslant 1)$ and $h(x) \in L^\infty(\Omega)$, $f(x, s)$ is asymptotically linear with respect to s at infinity. By a variant version of Mountain Pass Theorem, we prove that there exist at least two nonnegative solutions under suitable assumptions on $f(x, s)$. Our methods also work for the cases where $f(x, s)$ is superlinear or linear in s. © 2002 Elsevier Science (USA)

Key Words: elliptic problem; sublinear; superlinear; asymptotically linear; Mountain Pass Theorem.

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1. INTRODUCTION

In this paper, we study the following Dirichlet problem:

$$\begin{cases} -\Delta u = h(x)u^q + f(x,u), \\ 0 \leqslant u \in H_0^1(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$ $(N \geqslant 1)$, $0 < q < 1$ and $h(x)$, $f(x,s)$ satisfy the following conditions:

$(h1)$: $h(x) \in L^\infty(\Omega)$ and $h(x) \not\equiv 0$.

$(f1)$: $f(x,s) \in C(\bar{\Omega} \times \mathbf{R})$; $f(x,0) \equiv 0$; $f(x,s) \geqslant (\not\equiv) 0$ for all $s \geqslant 0, x \in \Omega$.

$(f2)$: $\lim_{s\to 0^+} \frac{f(x,s)}{s} = \mu \in [0, \lambda_1)$; $\lim_{s\to+\infty} \frac{f(x,s)}{s} = \ell \in (\lambda_1, +\infty]$ uniformly in $x \in \Omega$, where $\lambda_1 > 0$ is the first eigenvalue of $-\triangle$ in $H_0^1(\Omega)$, that is,

$$\lambda_1 = \inf\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2\,dx} : u \in H_0^1(\Omega), u \not\equiv 0 \right\}. \tag{1.2}$$

Moreover, if $\ell = +\infty$ we suppose as usual that $f(x,s)$ is subcritical, that is

$$\lim_{s\to+\infty} \frac{f(x,s)}{s^k} = 0 \qquad \text{uniformly in } x \in \Omega, \tag{1.3}$$

where k is some constant such that $k \in (1, (N+2)/(N-2))$ if $N \geqslant 3$ and $k \in (1, +\infty)$ if $N = 1, 2$.

Definition. We say that $u \in H_0^1(\Omega)$ is a *positive* (*nonnegative*) *weak solution* to problem (1.1) if $u > 0$ $(u \geqslant 0)$ a.e. on Ω and satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi\,dx = \int_\Omega h(x)u^q \varphi\,dx + \int_\Omega f(x,u)\varphi\,dx \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{1.4}$$

By assumption $(f1)$, we know that to seek a nonnegative weak solution of problem (1.1) is equivalent to finding a nonzero critical point of the following functional on $H_0^1(\Omega)$:

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{q+1}\int_\Omega h(x)(u^+)^{q+1}\,dx - \int_\Omega F(x,u^+)\,dx, \tag{1.5}$$

where $u^+ = \max\{0, u\}$, $F(x,u) = \int_0^u f(x,s)\,ds$. By $(f1)$ $(f2)$, it is easy to see that $I \in C^1(H_0^1(\Omega), \mathbf{R})$. By the strong maximum principle [15], the nonzero critical points of (1.5) are positive solutions to problem (1.1) if $h(x) \geqslant 0$.

Throughout this paper, the norm of $H_0^1(\Omega)$ is given by

$$\|u\| = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}} \qquad \text{for } u \in H_0^1(\Omega)$$

and we use $|\cdot|_p$ to denote the usual norm on $L^p(\Omega)$ for $1 \leqslant p \leqslant +\infty$.

Problem (1.1) was studied widely under various conditions on $h(x)$ and $f(x,s)$, see, for example, [9, 27, 31] where $q = 0$ and $f(x,s)$ is the Sobolev critical growth; [1–3, 16, 17] where $0 < q < 1$ and $f(x,s)$ is superlinear with critical or subcritical growth and [5–8, 10, 12, 18, 19, 24–26, 30] for $h(x) \equiv 0$ with Ω bounded or unbounded. Problem (1.1) was also studied by Wu and Yang in [29] for the case where $f(x,u) \equiv \lambda u - g(x,u)$, λ is a constant and $g(x,u) \not\equiv 0$ is sublinear globally in u. Moreover, Wang in [28] proved that problem (1.1) has infinitely many solutions $u_n \in H_0^1(\Omega)$ such that $|u_n|_\infty \overset{n}{\to} 0$, $I(u_n) < 0$ and $I(u_n) \overset{n}{\to} 0$ by assuming that $h(x) \equiv \lambda$ is a positive constant and $f(x,u)$ is odd for small u. To the authors' knowledge, it seems very few results on the case that $h(x)$ is not a constant and $f(x,s)$ is asymptotically linear in s at infinity, this case is different from the case in which $f(x,s)$ is superlinear or sublinear. Recently, by using variational methods and critical groups, Perera in [22] studied the existence of multiple solutions to problem (1.1) with $h(x) \equiv -\lambda$, $\lambda > 0$ is a constant and he proved that there exists $\lambda^* > 0$ such that problem (1.1) has at least 3–5 nontrivial solutions if $\lambda \in (0, \lambda^*)$ and $f(x,u) \equiv g(u) \in C^1$ satisfies the following conditions:

(g_1): $g(0) = 0$.

(g_2): $g'(0) > \lambda_1$, or

$(g_2)'$: $g'(0) > \lambda_k$, $k \geqslant 2$.

(g_3): $G(u) := \int_0^u g(t)\, dt \leqslant \frac{1}{2} a u^2$ for all u with $|u|$ large and $a < \lambda_1$, or

$(g_3)'$: $\beta := \lim_{u \to -\infty} \frac{g(u)}{u} < \lambda_1 \leqslant \lambda_k < a := \lim_{u \to +\infty} \frac{g(u)}{u} < \lambda_{k+1}$ for some $k \geqslant 1$.

(g_4): $G(u) \leqslant \frac{1}{2} \lambda_{k+1} u^2$ for all u and some $k \geqslant 2$,

where $0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots$ denote the eigenvalues of $-\triangle$ with zero Dirichlet boundary data.

In the present paper, we are concerned with problem (1.1) under the assumptions $(h1)$ $(f1)$ $(f2)$ which are different from that of papers mentioned above, for example, in our case, conditions (g_2) $(g_2)'$ (g_3) are not satisfied. Instead of applying some abstract theory, the method used in this paper is just based on a simple version of Mountain Pass Theorem, our method also works for the case where $f(x,s)$ is superlinear with respect to s at infinity or $f(x,s) \equiv \lambda s$ for some $\lambda > 0$, that is, the results of this paper cover all growing cases of $f(x,s)$ in s from linear to asymptotically linear and superlinear. Furthermore, our method is very hopeful to be used for much more general cases of problem (1.1), e.g., the analogue on unbounded domain. In this paper, we consider only nontrivial nonnegative solutions to problem (1.1), so $f(x, s) \geqslant 0$ for $s \geqslant 0$ is assumed.

The main results of this paper are as follows:

THEOREM 1.1. *Under conditions* $(h1)$ $(f1)$ *and* $(f2)$ *with* $\ell \in (\lambda_1, +\infty)$, *suppose further that there exists* $v \in H_0^1(\Omega)$ *such that*

$$(h2) \qquad \int_\Omega h(x)(v^+)^{q+1}\, dx > 0,$$

then there exists a constant $m = m(\mu, q, f, N, \Omega) > 0$ *such that for all* $h(x) \in L^\infty(\Omega)$ *with* $|h|_\infty < m$, *problem* (1.1) *has a solution* $u_1 \in H_0^1(\Omega)$, $u_1 \geqslant 0$ *and* $I(u_1) < 0$. *Moreover, if* $h(x) \geqslant 0$, *then* $u_1 > 0$ *a.e. in* Ω.

Remark 1.1. If $h(x) \geqslant (\not\equiv) 0$, it is easy to see that $(h2)$ is always satisfied.

THEOREM 1.2. *Under conditions* $(h1)$ $(f1)$ *and* $(f2)$ *with* $\ell \in (\lambda_1, +\infty)$, *there exists* $m = m(\mu, q, f, N, \Omega)$ *such that for all* $h \in L^\infty(\Omega)$ *with* $|h|_\infty < m$, *problem* (1.1) *has a nonnegative solution* $u_2 \in H_0^1(\Omega)$ *with* $I(u_2) > 0$ *and* $u_2 > 0$ *if* $h(x) \geqslant 0$.

COROLLARY 1.1. *Under the same conditions as Theorem* 1.2 *and if* $h(x) \geqslant (\not\equiv) 0$, *then there exists* $m > 0$ *and for all* $h(x) \in L^\infty(\Omega)$ *with* $|h|_\infty < m$, *problem* (1.1) *has at least two positive solutions* $u_1, u_2 \in H_0^1(\Omega)$ *such that* $I(u_1) < 0 < I(u_2)$.

This corollary is a straightforward conclusion of Theorems 1.1 and 1.2, by applying the strong maximum principle [15].

THEOREM 1.3. *Let* $(h1)$ $(f1)$, $(f2)$ *with* $\ell = +\infty$ *and* (1.3) *hold. Then*

(i) *If* $(h2)$ *holds and assume that for any* $\sigma \in (0, 1)$, $\tau > 0$ *small enough with* $\sigma > q + (1+q)\tau$ *such that*

$(f2)'$

$$\lim_{s \to +\infty} \frac{f(x, s)}{s} = +\infty, \quad \textit{but} \quad \lim_{s \to +\infty} \frac{f(x, s)}{s^{1+\tau}} = 0 \quad \textit{uniformly in } x \in \Omega.$$

and

(fF)

$$\lim_{s \to \infty} \frac{f(x, s)s - 2F(x, s)}{s^{1+\sigma}} = \eta > 0 \quad \textit{uniformly in } x \in \Omega,$$

where η can be $+\infty$, then there exists $m>0$ such that, for all $h\in L^{\infty}(\Omega)$ with $|h|_{\infty}<m$, problem (1.1) *has two nonnegative solutions $u_1,u_2\in H_0^1(\Omega)$ with $I(u_1)<0<I(u_2)$ and, u_1,u_2 are positive if $h(x)\geqslant 0$.*

(ii) *If $h(x)\leqslant(\not\equiv)0$ and for a.e. $x\in\Omega$,*

$(f3)$

$$\frac{f(x,s)}{s}\ \textit{is nondecreasing in}\ s>0,$$

then, problem (1.1) *has at least one nonnegative solution.*

Remark 1.2. Following the standard procedures of applying Mountain Pass Theorem [4], if the following well-known technical condition is satisfied, i.e. for some $\theta>0$ and $M>0$

$$(AR)\qquad 0<F(x,t)\leqslant\frac{1}{2+\theta}f(x,t)t\qquad\text{for all } x\in\Omega \text{ and } |t|\geqslant M,$$

then the conclusions of Theorem 1.3 are true. However, in the present paper our conditions $(f2)'$, (fF) or $(f3)$ in Theorem 1.3 are weaker, in some sense, than condition (AR), see the examples given below, in which all our conditions $(f1)$–$(f3)$ and (fF) are satisfied, but (AR). Clearly, if $\ell<\infty$ in $(f2)$ condition (AR) is impossible to be true, since condition (AR) implies that $f(x,s)$ in s has a behavior like $s^{1+\theta}$ at infinity. Moreover, if the second limit in condition $(f2)'$ is not zero, e.g. $\lim_{s\to+\infty}\frac{f(x,s)}{s^{1+\tau}}=\text{constant}>0$, then, roughly speaking, condition (AR) is satisfied, and also (fF) with $\eta=+\infty$. So we consider here only the case of $(f2)'$.

Example 1. For $\alpha>0$, $\ell>\lambda_1$, define

$$f(x,s)\equiv f(s):=\begin{cases}\dfrac{\ell s^{\alpha+1}}{1+s^{\alpha}} & \text{if } s\geqslant 0\\ 0 & \text{if } s\leqslant 0,\end{cases}$$

it is easy to see that $(f1)$, $(f2)$ with $\ell<\infty$, $(f3)$ are satisfied, and also (fF) with $\eta=+\infty$ if $\alpha\in(0,1-\sigma)$, but (AR) for any α. See e.g. [30].

Example 2. Define

$$f(x,s)\equiv f(s):=\begin{cases}s\ln(s+1) & \text{if } s\geqslant 0,\\ 0 & \text{if } s\leqslant 0,\end{cases}$$

then $(f1)$, $(f2)$ with $\ell=+\infty$, $(f2)'$ and (fF) with $\eta=+\infty$ are satisfied, but (AR). Indeed, if (AR) holds, then we have $\lim_{s\to+\infty}\frac{f(x,s)}{s^{1+\theta}}\neq 0$ which contradicts $(f2)'$ since τ in $(f2)'$ can be chosen small enough such that $\tau<\theta$.

This paper is organized as follows: in the first two sections we discuss the case where $f(x,s)$ is asymptotically linear with respect to s at infinity, i.e. $\ell < +\infty$ in $(f2)$. In Section 2, by using the Ekeland variational principle, we get a solution of problem (1.1) with negative energy (see Theorem 1.1). In Section 3, a second solution to problem (1.1) with positive energy is obtained by using a Mountain Pass Theorem for $\ell \in (\lambda_1, +\infty)$. The superlinear case, i.e. $\ell = +\infty$ in $(f2)$, is studied in Section 4 by using a similar method to Section 2. Moreover, a special case, i.e. $f(x,u) \equiv \lambda u$, is discussed in Section 5.

2. EXISTENCE OF A LOCAL MINIMUM

We begin this section by giving some auxiliary results about $f(x,s)$ based on conditions $(f1)$ and $(f2)$.

By $(f1)$, $(f2)$ with $\ell < +\infty$, and noticing that $f(x,s)/s^k \overset{s\to+\infty}{\longrightarrow} 0$ uniformly in $x \in \Omega$ for any fixed $k > 1$ with $k < \frac{N+2}{N-2}$ if $N \geqslant 3$ or $1 < k < +\infty$ if $N = 1,2$, then it is easy to see that for any $\varepsilon > 0$ there exists $C_\varepsilon = C(\varepsilon, k, f, \Omega) > 0$ such that

$$f(x,s) \leqslant (\mu + \varepsilon)s + C_\varepsilon s^k \qquad \text{for all } s \geqslant 0, \ x \in \Omega, \tag{2.1}$$

$$F(x,s) \leqslant \frac{\mu+\varepsilon}{2} s^2 + \frac{C_\varepsilon}{k+1} s^{k+1} \qquad \text{for all } s \geqslant 0, \ x \in \Omega. \tag{2.2}$$

Since $(f2)$, $\mu < \lambda_1$, we can find $\varepsilon_0 > 0$ with

$$\mu + \varepsilon_0 < \lambda_1, \tag{2.3}$$

and there is a $C_0 = C_0(k, \mu, f, \Omega) > 0$ such that

$$F(x,s) \leqslant \frac{\mu+\varepsilon_0}{2} s^2 + C_0 s^{k+1} \qquad \text{for all } s \geqslant 0, \ x \in \Omega. \tag{2.4}$$

Similarly, for $\ell = +\infty$, we have also (2.4) by using (1.3).

To prove Theorem 1.1, we need the following Ekeland variational principle:

PROPOSITION 2.1 (Ekeland's variational principle [13], Theorem 1.1 bis). *Let V be a complete metric space and $F : V \to \mathbf{R} \cup \{+\infty\}$ be lower semicontinuous, bounded from below. For any $\varepsilon > 0$, there is some point $v \in V$ with*

$$F(v) \leqslant \inf_V F + \varepsilon \qquad \textit{and} \qquad F(w) \geqslant F(v) - \varepsilon d(v,w) \qquad \textit{for all } w \in V. \quad \blacksquare$$

LEMMA 2.1. *If* $(h1)$ $(f1)$ *and* $(f2)$ *with* $\ell \in (\lambda_1, +\infty)$ *hold, then there exists* $m = m(u, q, f, N, \Omega) > 0$ *such that for all* $h \in L^\infty(\Omega)$ *with* $|h|_\infty < m$ *we have*

(i) *There exist* $\rho > 0$, $\eta > 0$ *such that*

$$I(u) \geqslant \eta > 0 \qquad \text{for all } u \in H_0^1(\Omega) \text{ with } \|u\| = \rho.$$

(ii) *There exists* $e \in H_0^1(\Omega)$ *with* $\|e\| > \rho$ *such that* $I(e) < 0$.

(iii) *If* $\ell = +\infty$ *in* $(f2)$ *and,* $(f3)$ *holds, then the conclusions of* (i) *and* (ii) *still hold.*

Proof. (i) Since $h(x) \leqslant |h(x)| \leqslant |h|_\infty$, noticing the definition of I given in (1.5), it follows from (2.4) and Sobolev's embedding that

$$\begin{aligned} I(u) &\geqslant \frac{1}{2}\|u\|^2 - \frac{|h|_\infty}{q+1}\int_\Omega (u^+)^{q+1}\,dx - \frac{\mu+\varepsilon_0}{2}\int_\Omega u^2\,dx - C_0\int_\Omega |u|^{k+1}\,dx \\ &\geqslant C_1\|u\|^2 - C_2|h|_\infty\|u\|^{q+1} - C_3\|u\|^{k+1} \\ &\geqslant [C_1 - |h|_\infty C_2\|u\|^{q-1} - C_3\|u\|^{k-1}]\|u\|^2, \end{aligned} \tag{2.5}$$

where $C_1 = \frac{1}{2}(1 - \frac{\mu+\varepsilon_0}{\lambda_1}) > 0$ by (2.3), $C_2 = C_2(q, N, \Omega) > 0$ and $C_3 = C_3(C_0, k, N, \Omega)$. Motivated by [16], we let

$$g(t) = C_2|h|_\infty t^{q-1} + C_3 t^{k-1} \qquad \text{for } t \geqslant 0,$$

where $q \in (0,1)$ and $k \in [1, \frac{N+2}{N-2})$ if $N \geqslant 3$, or $k \in [1, +\infty)$ if $1 \leqslant N < 3$. Clearly,

$$g'(t) = C_2(q-1)|h|_\infty t^{q-2} + C_3(k-1)t^{k-2}.$$

By $g'(t_0) = 0$, we have

$$t_0 = [C_4|h|_\infty]^{1/(k-q)}, \qquad 0 < q < 1 < k,$$

where $C_4 = \frac{(1-q)C_2}{C_3(k-1)}$, and then

$$g(t_0) = C_2|h|_\infty(C_4|h|_\infty)^{\frac{q-1}{k-q}} + C_3(C_4|h|_\infty)^{\frac{k-1}{k-q}} = C_5|h|_\infty^{\frac{k-1}{k-q}},$$

where $C_5 = C_5(q, k, \mu, f, N, \Omega) = C_2 C_4^{\frac{q-1}{k-q}} + C_3 C_4^{\frac{k-1}{k-q}}$, and $\frac{k-1}{k-q} > 0$ since $0 < q < 1 < k$. Thus, for any fixed $k > 1$, there exists $m = m(\mu, q, f, N, \Omega) > 0$ such that $g(t_0) < C_1$ if $|h|_\infty < m$. Then, if $|h|_\infty < m$ and taking $\rho = t_0$, it follows from (2.5) we see that (i) is proved.

(ii) Since $\lambda_1 < \ell < +\infty$, by $(f2)$ it is easy to see that

$$\lim_{s\to+\infty} \frac{F(x,s)}{s^2} = \frac{\ell}{2} > \frac{\lambda_1}{2} \qquad \text{uniformly in } x \in \Omega,$$

then, for $s > 0$ large enough we can find some $\tau > 0$ such that

$$\frac{F(x,s)}{s^2} \geqslant \frac{\ell - \tau}{2} > \frac{\lambda_1}{2} \qquad \text{uniformly in } x \in \Omega. \tag{2.6}$$

Let $\varphi_1 > 0$ be a λ_1-eigenfunction, for $t > 0$ large enough and noticing $0 < q < 1$, we have

$$\begin{aligned} I(t\varphi_1) &= \frac{t^2}{2}\|\varphi_1\|^2 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}\,dx - \int_\Omega F(x, t\varphi_1)\,dx \\ &\leqslant \frac{t^2}{2}\|\varphi_1\|^2 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}\,dx - \frac{t^2}{2}\int_\Omega (\ell-\tau)\varphi_1^2\,dx \\ &= \frac{t^2}{2}\int_\Omega (\lambda_1 - \ell + \tau)\varphi_1^2\,dx - \frac{t^{q+1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}\,dx \\ &< 0, \quad \text{by (2.6) and } t > 0 \text{ large.} \end{aligned}$$

So, for $t_0 > 0$ large enough and choosing $e = t_0\varphi_1$, then (ii) is proved.

(iii) Noting $(f2)$ and (1.3), (2.4) is also true, then same as the proof of part (i) we see that the conclusion of part (i) still holds for the case of $\ell = +\infty$.

On the other hand, by standard regularity result we know that λ_1-eigenfunction $\varphi_1 \in C(\bar{\Omega})$, then there exist $\Omega_0 \subset \bar{\Omega}_0 \subset\subset \Omega$ with $m\Omega_0 > 0$ and some $\alpha > 0$ such that $\min_{\bar{\Omega}_0} \varphi_1(x) \geqslant \alpha > 0$. Hence, $t\varphi_1(x) \overset{t\to+\infty}{\to} +\infty$ uniformly in $\bar{\Omega}_0$. By $(f3)$ we see that $0 \leqslant 2F(x,t) \leqslant f(x,t)t$ for all $x \in \Omega$ and $t \geqslant 0$, then $F(x,t)/t^2$ is nondecreasing in $t > 0$. Noticing $\ell \equiv +\infty$ in $(f2)$, then for any $x \in \Omega_0$ and $t > 0, \frac{F(x,t\varphi_1)}{t^2\varphi_1^2} \geqslant \frac{F(x,t\alpha)}{t^2\alpha^2} \overset{r\to+\infty}{\to} +\infty$ uniformly in $x \in \Omega_0$, and so for any $K > 0$, there exists $T = T(\alpha, K) > 0$ such that

$$\frac{F(x, t\varphi_1)}{t^2\varphi_1^2} \geqslant K > 0 \qquad \text{for all } t \geqslant T,\ x \in \bar{\Omega}_0.$$

Therefore, if we choose $K > 0$ large enough and for $t > T$, we have

$$\begin{aligned}\frac{I(t\varphi_1)}{t^2} &\leqslant \frac{1}{2}\int_\Omega |\nabla\varphi_1|^2\,dx - \int_{\Omega_0} \frac{F(x,\,t\varphi_1)}{t^2\varphi_1^2}\,\varphi_1^2\,dx - \frac{t^{q-1}}{q+1}\int_\Omega h(x)\varphi_1(x)^{q+1}\,dx\\ &\leqslant \frac{1}{2}\int_\Omega |\nabla\varphi_1|^2\,dx - K\int_{\Omega_0}\varphi_1^2\,dx\\ &\leqslant \frac{1}{2}\int_\Omega |\nabla\varphi_1|^2\,dx - \alpha^2 K\, m\Omega_0 < 0.\end{aligned}$$

This means that part (ii) is also proved for $\ell \equiv +\infty$. ∎

Proof of Theorem 1.1. Similar to [20], for $\rho > 0$ given by Lemma 2.1(i), define

$$\bar{B}_\rho = \{u \in H_0^1(\Omega) : \|u\| \leqslant \rho\}, \qquad \partial B_\rho = \{u \in H_0^1(\Omega) : \|u\| = \rho\}$$

and $\bar{B}_\rho$ is a complete metric space with the distance

$$\mathrm{dist}(u, v) = \|u - v\| \qquad \text{for } u, v \in \bar{B}_\rho.$$

By Lemma 2.1, we know that

$$I(u)|_{\partial B_\rho} \geqslant \eta > 0. \tag{2.7}$$

Clearly, $I \in C^1(\bar{B}_\rho, \mathbf{R})$, hence I is lower semicontinuous and bounded from below on $\bar{B}_\rho$. Let

$$c_1 = \inf\{I(u) : u \in \bar{B}_\rho\}. \tag{2.8}$$

We claim that

$$c_1 < 0. \tag{2.9}$$

Indeed, let $v \in C_0^\infty(\Omega)$ be given by $(h2)$, i.e. $\int_\Omega h(x)(v^+)^{q+1}(x)\,dx > 0$, then for $t > 0$ small, we have

$$\begin{aligned}I(tv) &= \frac{t^2}{2}\int_\Omega |\nabla v|^2\,dx - \frac{t^{q+1}}{q+1}\int_\Omega h(x)(v^+)^{q+1}\,dx - \int_\Omega F(x,\,tv^+)\,dx\\ &\leqslant \frac{t^2}{2}\int_\Omega |\nabla v|^2\,dx - \frac{t^{q+1}}{q+1}\int_\Omega h(x)|v^+|^{q+1}\,dx - \frac{\mu t^2}{4}\int_\Omega |v^+|^2\,dx \qquad \text{by } (f2)\\ &< 0, \qquad \text{if } t > 0 \text{ small enough.}\end{aligned}$$

So, (2.9) is proved.

By Proposition 2.1, for any $k > 0$, there is a u_k such that

$$c_1 \leqslant I(u_k) \leqslant c_1 + \frac{1}{k}, \tag{2.10}$$

$$I(w) \geqslant I(u_k) - \frac{1}{k}\,\|u_k - w\| \qquad \text{for any } w \in \bar{B}_\rho. \tag{2.11}$$

Then, $\|u_k\| < \rho$ for $k \geqslant 1$ large enough. Otherwise, if $\|u_k\| = \rho$ for infinitely many k, without loss of generality, we may assume that $\|u_k\| = \rho$ for all $k \geqslant 1$, and it follows from (2.7) that

$$I(u_k) \geqslant \eta > 0$$

letting $k \to \infty$ and combining (2.10) we see that $0 > c_1 \geqslant \eta > 0$, this is a contradiction.

We prove now that $I'(u_k) \xrightarrow{n} 0$ in $H_0^{-1}(\Omega)$. In fact, for any $u \in H_0^1(\Omega)$ with $\|u\| = 1$, let $w_k = u_k + tu$ and for any fixed $k \geqslant 1$, we have $\|w_k\| \leqslant \|u_k\| + t < \rho$ if $t > 0$ small enough. So, it follows from (2.11) that

$$I(u_k + tu) \geqslant I(u_k) - \frac{t}{k}\,\|u\|,$$

that is,

$$\frac{I(u_k + tu) - I(u_k)}{t} \geqslant -\frac{1}{k}\,\|u\| = -\frac{1}{k}.$$

Letting $t \to 0$, we see that $\langle I'(u_k), u\rangle \geqslant -\frac{1}{k}$, and this gives

$$|\langle I'(u_k), u\rangle| < \frac{1}{k} \qquad \text{for any } u \in H_0^1(\Omega) \text{ with } \|u\| = 1.$$

So, $I'(u_k) \xrightarrow{n} 0$ in $H_0^{-1}(\Omega)$, and by (2.10), $I(u_k) \to^n c_1 < 0$. Hence, by the compactness of Sobolev embedding and a standard procedure, we see that there exists $u_1 \in H_0^1(\Omega)$ such that $I'(u_1) = 0$, that is, u_1 is a nonnegative weak solution of problem (1.1) and $I(u_1) = c_1 < 0$. Moreover, if $h(x) \geqslant 0$, the strong maximum principle [15] implies that $u_1 > 0$ a.e. in Ω, and the proof of Theorem 1.1 is completed. ∎

3. EXISTENCE OF A MOUNTAIN PASS-TYPE SOLUTION

In this section, we use a variant version of Mountain Pass Theorem to get a nonzero critical point of functional I, this theorem is used also in [11] and its proof can be found in [14, 23], let us recall first this theorem.

PROPOSITION 3.1 (Mountain Pass Theorem). *Let E be a real Banach space with its dual space E^* and suppose that $I \in C^1(E, \mathbf{R})$ satisfy the condition*

$$\max\{I(0), I(e)\} \leqslant \mu < \eta \leqslant \inf_{\|u\|=\rho} I(u)$$

for some $\mu < \eta$, $\rho > 0$ and $e \in E$ with $\|e\| > \rho$. Let $c \geqslant \eta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leqslant \tau \leqslant 1} I(\gamma(\tau)),$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ is the set of continuous paths joining 0 and e. Then, there exists a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \overset{n}{\to} c \geqslant \eta \qquad \textit{and} \qquad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \overset{n}{\to} 0. \quad ∎$$

Proof of Theorem 1.2. Let ρ, η and e be given in Lemma 2.1, applying Proposition 3.1 with $\mu = 0$, $E = H_0^1(\Omega)$, and for c defined as in Proposition 3.1, then there exists a sequence $\{u_n\} \subset H_0^1(\Omega)$ such that

$$I(u_n) \overset{n}{\to} c > 0, \qquad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \overset{n}{\to} 0.$$

This implies that

$$\frac{1}{2}\|u_n\|^2 - \frac{1}{q+1}\int_\Omega h(x)(u_n^+)^{q+1}\,dx - \int_\Omega F(x, u_n^+)\,dx = c + o(1), \tag{3.1}$$

$$\int_\Omega \nabla u_n \cdot \nabla\varphi\,dx - \int_\Omega h(x)(u_n^+)^q\varphi\,dx - \int_\Omega f(x, u_n^+)\varphi\,dx = o(1) \quad \text{for } \varphi \in H_0^1(\Omega), \tag{3.2}$$

$$\int_\Omega |\nabla u_n|^2\,dx - \int_\Omega h(x)(u_n^+)^{q+1}\,dx - \int_\Omega f(x, u_n^+)u_n\,dx = o(1). \tag{3.3}$$

By the compactness of Sobolev embedding and the standard procedures, we know that, if $\{u_n\}$ is bounded in $H_0^1(\Omega)$, there exists $u_2 \in H_0^1(\Omega)$ such that $I'(u_2) = 0$, $I(u_2) = c > 0$ and u_2 is a nonnegative weak solution of problem (1.1), which is positive if $h(x) \geqslant 0$ by the strong maximum principle. Moreover u_2 is different from the solution u_1 obtained in Theorem 1.1 since $I(u_1) = c_1 < 0$. So, to prove Theorem 1.2, we need only show that $\{u_n\}$ given by (3.1)–(3.3) is bounded in $H_0^1(\Omega)$.

We turn now to showing that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. By contradiction, we suppose that $\|u_n\| \overset{n}{\to} \infty$, and we set

$$w_n = \frac{u_n}{\|u_n\|}. \tag{3.4}$$

Clearly, w_n is bounded in $H_0^1(\Omega)$, and we may suppose, for some subsequence of $\{w_n\}$ and $w \in H_0^1(\Omega)$, that

$$\begin{cases} w_n \overset{n}{\rightharpoonup} w \quad \text{weakly in } H_0^1(\Omega), \quad w_n \overset{n}{\to} w \quad \text{a.e. in } \Omega, \\ \qquad\qquad w_n \overset{n}{\to} w \quad \text{strongly in } L^r(\Omega), \end{cases} \tag{3.5}$$

where $1 \leqslant r < 2^* = \frac{2N}{N-2}$ if $N \geqslant 3$ and $r \in (1, +\infty)$ if $N = 1, 2$.

Similarly, $w_n^+ = \frac{u_n^+}{\|u_n\|}$ satisfying

$$\begin{cases} w_n^+ \overset{n}{\rightharpoonup} w^+ \quad \text{weakly in } H_0^1(\Omega), \quad w_n^+ \overset{n}{\to} w^+ \quad \text{a.e. in } \Omega, \\ w_n \overset{n}{\to} w^+ \quad \text{strongly in } L^r(\Omega). \end{cases} \tag{3.6}$$

We first claim that

$$w \not\equiv 0.$$

Indeed, if $w \equiv 0$, then by $(f1)$ and (3.6) we have

$$\lim_{n\to\infty} \int_\Omega h(x)(w_n^+)^{q+1}\,dx = 0,$$
$$\lim_{n\to\infty} \int_\Omega F(x, w_n^+(x))\,dx = \lim_{n\to\infty} \int_\Omega f(x, w_n^+(x))w_n(x)\,dx = 0.$$

Multiplying (3.3) by $\frac{1}{\|u_n\|^2}$, and noticing (3.4) we see that

$$\|w_n\|^2 - \frac{1}{\|u_n\|^{1-q}} \int_\Omega h(x)(w_n^+)^{q+1}\,dx - \int_\Omega p(x, u_n)(w_n^+)^2\,dx = o(1), \tag{3.7}$$

where

$$p(x, s) = \begin{cases} \dfrac{f(x, s^+)}{s^+} & \text{if } s > 0, \\ 0 & \text{if } s \leqslant 0. \end{cases}$$

and $p(x, s) = p(x, s^+) \geqslant 0$. Moreover, it follows from $(f1)$ and $(f2)$ with $\ell \in (\lambda_1, +\infty)$ that there exists a constant $M > 0$ such that

$$\left| \frac{f(x, s^+)}{s^+} \right| \leqslant M \qquad \text{for all } x \in \Omega \text{ and } s \in \mathbf{R}. \tag{3.8}$$

Noting $\|u_n\| \overset{n}{\to} +\infty$, (3.7) yields

$$\|w_n\|^2 - \int_\Omega p(x, u_n^+)(w_n^+)^2\,dx = o(1),$$

but (3.6) and (3.8) imply that

$$\left| \int_\Omega p(x, u_n^+)(w_n^+)^2 \, dx \right| \leqslant M \int_\Omega (w_n^+)^2 \, dx \xrightarrow{n} 0.$$

So, $\|w_n\|^2 \xrightarrow{n} 0$, this contradicts that $\|w_n\| = 1$.

Next, we claim that

$$w(x) > 0 \qquad \text{a.e. in } \Omega.$$

By (3.2), we know that, for any $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \nabla w_n \cdot \nabla \varphi \, dx - \frac{1}{\|u_n\|^{1-q}} \int_\Omega h(x)(w_n^+)^q \varphi \, dx$$
$$- \int_\Omega p(x, u_n^+) w_n \varphi \, dx = o(1). \tag{3.9}$$

By (3.8), there exists some $v \in L^2(\Omega)$ such that

$$p(x, u_n) \overset{n}{\rightharpoonup} v(x) \qquad \text{weakly in } L^2(\Omega) \text{ and } 0 \leqslant v(x) \leqslant M \text{ a.e. in } \Omega,$$

this and (3.6) imply that, for any $\varphi \in H_0^1(\Omega)$,

$$\int_\Omega \frac{f(x, u_n^+)}{u_n^+} w_n^+ \varphi \, dx = \int_\Omega p(x, u_n) w_n^+(x) \varphi \, dx \xrightarrow{n} \int_\Omega v(x) w^+ \varphi \, dx.$$

So, combining (3.9) we get

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx = \int_\Omega v(x) w^+ \varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{3.10}$$

Let $\varphi = w^- = \min\{0, w\}$ in (3.10), it is easy to see that $\int_\Omega |\nabla w^-|^2 \, dx = 0$, this means that $w^- = 0$ a.e. in Ω, and then $w \equiv w^+ \geqslant 0$. By (3.10), we know that $w \geqslant 0$ is a weak solution to the following problem:

$$- \triangle w = v(x) w^+ \geqslant 0,$$

then it follows from the strong maximum principle for weak solution (see [15]) that $w(x) > 0$ a.e. in Ω.

Finally, we turn to showing that $w(> 0)$ satisfies the following identity:

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx = \ell \int_\Omega w \varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega). \tag{3.11}$$

By (3.4) and (3.6), we see that

$$w_n(x) = \frac{u_n}{\|u_n\|} \xrightarrow{n} w(x) > 0 \qquad \text{a.e. in } \Omega,$$

but by our assumption that $\|u_n\| \xrightarrow{n} +\infty$, then we must have $u_n(x) \xrightarrow{n} +\infty$ a.e. in Ω, and it follows from $(f2)$ that $p(x, u_n^+) = \frac{f(x,u_n^+)}{u_n^+} \xrightarrow{n} \ell$. Hence $v(x) \equiv \ell$ in (3.10), that is, (3.11) holds. However, (3.11) contradicts that $\ell > \lambda_1$ since λ_1 is the smallest eigenvalue of $-\Delta$, so our assumption that $\|u_n\| \xrightarrow{n} +\infty$ is false, that is, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and the proof of Theorem 1.2 is completed. ∎

4. THE CASE OF $\ell = +\infty$

If $\ell = +\infty$ in $(f2)$, this is a very common case that $f(x, s)$ is called superlinear with respect to s at infinity, but to deal with this kind of problem, as it is well-known, condition (AR) is usually required. The main aim of this section is to prove that problem (1.1) still has solutions if $\ell = +\infty$ but without assuming condition (AR) (see Theorem 1.3). In order to prove Theorem 1.3, we need the following lemma.

LEMMA 4.1. *For I defined by* (1.5), *if* $(f1)$ $(f2)$ $(f3)$ *hold and there exists* $\{u_n\} \subset H_0^1(\Omega)$ *satisfies*

$$\langle I'(u_n), u_n \rangle \xrightarrow{n} 0,$$

then, for any $t > 0$, *by extracting a suitable subsequence, we have*

$$I(tu_n) \leqslant \frac{t^2+1}{2n} + \left[\frac{t^2}{2} - \frac{t^{q+1}}{1+q}\right] \int_\Omega h(x)(u_n^+)^{q+1}\,dx + I(u_n) \qquad \textit{if } h(x) \leqslant 0.$$

Proof. The main ideal of this proof is essentially due to [26] where $h(x) \equiv 0$. For the sake of completeness, we give its proof. By the assumption, we may assume that there is a subsequence (still denoted by $\{u_n\}$) such that for all $n \geqslant 1$, $-\frac{1}{n} \leqslant \langle I'(u_n), u_n \rangle \leqslant \frac{1}{n}$, that is,

$$\begin{aligned} -\frac{1}{n} + \int_\Omega f(x, u_n^+)u_n^+\,dx \leqslant \|u_n\|^2 - \int_\Omega h(x)(u_n^+)^{q+1}\,dx \\ \leqslant \frac{1}{n} + \int_\Omega f(x, u_n^+)u_n^+\,dx. \end{aligned} \tag{4.1}$$

Then, for any $t > 0$, by using the right side inequality of (4.1), we have

$$\begin{aligned} I(tu_n) &= \frac{t^2}{2}\|u_n\|^2 - \frac{t^{q+1}}{q+1}\int_\Omega h(x)(u_n^+)^{q+1}\,dx - \int_\Omega F(x, tu_n^+)\,dx \\ &\leqslant \frac{t^2}{2n} + \left[\frac{t^2}{2} - \frac{t^{q+1}}{q+1}\right]\int_\Omega h(x)(u_n^+)^{q+1}\,dx \\ &\quad + \int_\Omega \left[\frac{t^2}{2} f(x, u_n^+)u_n^+ - F(x, tu_n^+)\right] dx. \end{aligned} \tag{4.2}$$

For $s \geqslant 0$, $t \geqslant 0$, let $g(t) = \frac{1}{2}t^2 f(x, s)s - F(x, ts)$, by $(f3)$ it is easy to see that

$$g'(t) = f(x, s)ts - f(x, ts)s = \begin{cases} \geqslant 0 & \text{if } t \leqslant 1, \\ \leqslant 0 & \text{if } t \geqslant 1, \end{cases}$$

this means that $g(t) \leqslant g(1)$, for all $t > 0$. So, it follows from (4.2) that

$$\begin{aligned} I(tu_n) \leqslant \frac{t^2}{2n} + \left[\frac{t^2}{2} - \frac{t^{q+1}}{q+1}\right] \int_\Omega h(x)(u_n^+)^{q+1}\,dx \\ + \int_\Omega \left[\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right] dx. \end{aligned} \tag{4.3}$$

On the other hand, by the left side inequality of (4.1) we know that

$$\begin{aligned} I(u_n) \geqslant & -\frac{1}{2n} + \left[\frac{1}{2} - \frac{1}{q+1}\right] \int_\Omega h(x)(u_n^+)^{q+1}\,dx \\ & + \int_\Omega \left[\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right] dx \\ \geqslant & -\frac{1}{2n} + \int_\Omega \left[\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right] dx \qquad \text{if } h(x) \leqslant 0 \end{aligned}$$

that is, for $h(x) \leqslant 0$ we have that

$$\int_\Omega \left[\frac{1}{2} f(x, u_n^+)u_n^+ - F(x, u_n^+)\right] dx \leqslant \frac{1}{2n} + I(u_n). \tag{4.4}$$

Therefore, combining (4.3) and (4.4) we know that the proof of Lemma 4.1 is completed. ∎

Proof of Theorem 1.3. *Proof of* (i): By Lemma 2.1(iii) and the proof of Theorem 1.1, we know that problem (1.1) has a nonnegative solution $u_1 \in H_0^1(\Omega)$ with $I(u_1) < 0$.

Now we turn to showing that (1.1) has a Mountain Pass-type solution u_2. For this purpose, similar to the proof of Theorem 1.2, it follows from Lemma 2.1(iii) and Proposition 3.1 that there exists a sequence $\{u_n\} \in H_0^1(\Omega)$ such that (3.1)–(3.3) hold, then we need only show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, but to prove this conclusion we cannot follow the same way as that used in the proof of Theorem 1.2 since $\ell = \infty$ now.

By condition $(f2)'$, we see that, for any given $\varepsilon > 0$, there exists $T_1 > 0$ such that

$$f(x, t) \leqslant \varepsilon t^{1+\tau} \qquad \text{for all } t \geqslant T_1 \text{ and for a.e. } x \in \Omega, \tag{4.5}$$

and it follows from (fF) that there exists $T_2 > 0$ such that

$$f(x, t)t - 2F(x, t) \geqslant \frac{\eta}{2} t^{1+\sigma} > 0 \qquad \text{for all } t \geqslant T_2 \text{ and a.e. } x \in \Omega \quad (4.6)$$

Taking $T = \max\{T_1, T_2\}$ and, for each $n \geqslant 1$, setting

$$A_n = \{x \in \Omega : |u_n(x)| \geqslant T\}, \qquad B_n = \{x \in \Omega : |u_n(x)| \leqslant T\}.$$

Then by $(f1)$, for $\{u_n\}$ given in (3.1)–(3.3), there is some $C_0 = C_0(T) > 0$ such that

$$\begin{aligned} -C_0 \leqslant \int_{B_n} [f(x, u_n)u_n - 2F(x, u_n)]\, dx \leqslant C_0, \\ f(x, u_n)u_n - 2F(x, u_n) \geqslant 0 \qquad \text{in } A_n. \end{aligned} \quad (4.7)$$

For the above $T > 0$, it follows from (3.1), (3.3), (4.7) and (4.6) that

$$\begin{aligned} \left(\frac{2}{q+1} - 1\right) \int_\Omega |u_n^+|^{q+1}\, dx + 2c + o(1) &= \int_\Omega [f(x, u_n^+)u_n^+ - 2F(x, u_n^+)]\, dx \\ &\geqslant \int_{A_n} [f(x, u_n^+)u_n^+ - 2F(x, u_n^+)]\, dx - C_0 \\ &\geqslant \frac{\eta}{2} \int_{A_n} |u_n^+|^{1+\sigma}\, dx - C_0, \end{aligned}$$

where $|A_n|$ denotes the measure of A_n. This implies that there exist $C_1 = C_1(C_0, c, \eta) > 0$, $C_2 = C_2(\eta, q) > 0$ such that

$$\int_{A_n} |u_n^+|^{1+\sigma}\, dx \leqslant C_1 + C_2 \|u_n\|^{q+1} + o(1). \quad (4.8)$$

On the other hand, fixing $m = \frac{1+q}{q} > 2$ since $q \in (0, 1)$. By (3.1) and (3.3) we get

$$\begin{aligned} &\frac{1-q}{2(1+q)} \|u_n\|^2 - \frac{1-q}{q+1} \int_\Omega h(x)|u_n^+|^{q+1}\, dx - \int_\Omega \left[F(x, u_n^+) - \frac{1}{m} f(x, u_n^+)u_n^+\right] dx \\ &\quad = c + o(1), \end{aligned} \quad (4.9)$$

and noting that, by $(f1)$ and the definition of B_n, there exists $C_3 > 0$ such that

$$\left| \int_{B_n} \left[F(x, u_n^+) - \frac{1}{m} f(x, u_n^+)u_n^+\right] dx \right| \leqslant C_3.$$

Then, it follows from (4.7)–(4.9) that

$$\begin{aligned}
\frac{1-q}{2(1+q)}\|u_n\|^2 + o(1) &\leqslant c + C_3 + \int_{A_n}\left[F(x,u_n^+) - \frac{1}{m}f(x,u_n^+)u_n^+\right]dx \\
&\quad + \frac{1-q}{1+q}\int_\Omega h(x)|u_n^+|^{q+1}\,dx \\
&\leqslant c + C_3 + \int_{A_n}\left[\frac{1}{2}f(x,u_n^+)u_n - \frac{q}{1+q}f(x,u_n^+)u_n\right]dx \\
&\quad + \frac{1-q}{1+q}\int_\Omega h(x)|u_n^+|^{q+1}\,dx \\
&= c + C_3 + \frac{1-q}{2(1+q)}\int_{A_n} f(x,u_n^+)u_n^+\,dx \\
&\quad + \frac{C(1-q)}{1+q}|h(x)|_\infty\|u_n^+\|^{q+1}, \qquad \text{by Sobolev embedding} \\
&\leqslant c + C_3 + \int_{A_n}|u_n^+|^{2+\tau}\,dx + C|h(x)|_\infty\|u_n^+\|^{q+1}, \\
&\quad \text{by taking } \varepsilon = \frac{2(1+q)}{1-q} \text{ in (4.5)} \\
&\leqslant c + C_3 + \left(\int_{A_n}|u_n^+|^{1+\sigma}\,dx\right)^{\frac{1+\tau}{1+\sigma}}\left(\int_{A_n}|u_n^+|^{\frac{1+\sigma}{\sigma-\tau}}\,dx\right)^{\frac{\sigma-\tau}{\sigma+1}} \\
&\quad + C|h(x)|_\infty\|u_n^+\|^{q+1}, \qquad \text{by Hölder inequality} \\
&\leqslant c + C_3 + C\left(\int_{A_n}|u_n^+|^{1+\sigma}\,dx\right)^{\frac{1+\tau}{1+\sigma}}\|u_n\| \\
&\quad + C|h(x)|_\infty\|u_n^+\|^{q+1}, \qquad \text{by Sobolev imbedding} \\
&= c + C_3 + \frac{1-q}{4(1+q)}\|u_n\|^2 + \frac{(1+q)C^2}{1-q}\left(\int_{A_n}|u_n^+|^{1+\sigma}\,dx\right)^{\frac{2(1+\tau)}{1+\sigma}} \\
&\quad + C|h(x)|_\infty\|u_n^+\|^{q+1}, \qquad \text{by Young's inequality} \\
&\leqslant C|h(x)|_\infty\|u_n\|^{q+1} + c + C_3 + \frac{1-q}{4(1+q)}\|u_n\|^2 \\
&\quad + \frac{(1+q)C^2}{1-q}(C_2\|u_n\|^{q+1} + C_1)^{\frac{2(1+\tau)}{1+\sigma}}.
\end{aligned}$$

This implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$ since $0<q<1$ and $\frac{2(1+q)(1+\tau)}{1+\sigma}<2$ by noting that $q+(1+q)\tau<\sigma<1$.

Proof of (ii): In this case, we also use Proposition 3.1 to prove that problem (1.1) has a Mountain Pass-type solution. Similar to the proof of part (i), we show only that the sequence $\{u_n\}$ given by (3.1)–(3.3) is bounded in $H_0^1(\Omega)$.

For this purpose, we argue by contradiction. Suppose that $\|u_n\| \overset{n}{\to} \infty$ and, for $c > 0$ given in (3.1) we set

$$t_n = \frac{2\sqrt{c}}{\|u_n\|}, \qquad w_n = t_n u_n = \frac{2\sqrt{c}u_n}{\|u_n\|}. \tag{4.10}$$

Clearly, w_n is bounded in $H_0^1(\Omega)$. By extracting a subsequence and similar to (3.5) and (3.6), we may suppose that, for the same r given by (3.5),

$$\begin{cases} w_n \overset{n}{\rightharpoonup} w, \quad w_n^+ \overset{n}{\rightharpoonup} w^+ & \text{weakly in } H_0^1(\Omega), \\ w_n \overset{n}{\to} w, \quad w_n^+ \overset{n}{\to} w^+ & \text{a.e. in } \Omega, \\ w_n \overset{n}{\to} w, \quad w_n^+ \overset{n}{\to} w^+ & \text{strongly in } L^r(\Omega). \end{cases} \tag{4.11}$$

We claim that

$$w^+ \not\equiv 0.$$

In fact, if $w^+ \equiv 0$, then by (4.11) we have $w_n^+ \overset{n}{\to} 0$ in $L^2(\Omega)$ and $L^{q+1}(\Omega)$. Hence,

$$\lim_{n\to\infty} \int_\Omega h(x)(w_n^+)^{q+1}\, dx = 0; \qquad \lim_{n\to\infty} \int_\Omega F(x, w_n^+(x))\, dx = 0,$$

therefore

$$I(w_n) = \frac{1}{2}\|w_n\|^2 - \frac{1}{q+1}\int_\Omega h(x)(w_n^+)^{q+1}\, dx - \int_\Omega F(x, w_n^+)\, dx \tag{4.12}$$

$$= 2c - o(1). \tag{4.13}$$

On the other hand, by (4.10), $t_n \to 0$ as $n \to +\infty$ since $\|u_n\| \overset{n}{\to} \infty$. Hence, it follows from Lemma 4.1 with t replaced by t_n that

$$\begin{aligned} I(w_n) = I(tu_n) &\leqslant \frac{t_n^2+1}{2n} + \left[\frac{t_n^2}{2} - \frac{t_n^{q+1}}{1+q}\right]\int_\Omega h(x)(u_n^+)^{q+1}\, dx + I(u_n) \\ &= \frac{t_n^2}{2n} + \left[\frac{t_n^{1-q}}{2} - \frac{1}{1+q}\right]\int_\Omega h(x)(w_n^+)^{q+1}\, dx + I(u_n) \overset{n}{\to} c, \end{aligned}$$

that is, $I(w_n) \leqslant c + o(1)$, which contradicts (4.13). Hence, $w^+ \not\equiv 0$.

Now, we split Ω into two parts as follows:

$$\Omega_1 = \{x \in \Omega : w^+(x) = 0\}, \qquad \Omega_2 = \{x \in \Omega : w^+(x) > 0\}.$$

By (4.10) and (4.11), we see that $u_n^+ \overset{n}{\to} +\infty$ a.e. in Ω_2. Using (3.2) and noticing (4.10), the same as that of Section 2 we have (3.9). Taking $\varphi = w^+$ in (3.9) we get

$$\int_\Omega \nabla w_n \cdot \nabla w^+ \, dx - \frac{1}{||u_n||^{1-q}} \int_\Omega h(x)(w_n^+)^q w^+ \, dx - \int_\Omega p_n(x, u_n) w_n^+ w^+ \, dx = o(1), \tag{4.14}$$

where

$$p_n(x, u_n) = \begin{cases} \frac{f(x, u_n(x))}{u_n(x)} & \text{for } x \in \Omega \text{ with } u_n(x) \geqslant 0, \\ 0 & \text{for } x \in \Omega \text{ with } u_n(x) \leqslant 0, \end{cases}$$

therefore

$$\int_\Omega |\nabla w^+|^2 \, dx = \lim_{n \to +\infty} \int_\Omega \frac{f(x, u_n^+)}{u_n^+} w_n^+ w^+ \, dx \geqslant \lim_{n \to +\infty} \int_{\Omega_2} \frac{f(x, u_n^+)}{u_n^+} w_n^+ w^+ \, dx \geqslant \int_{\Omega_2} \lim_{n \to +\infty} \left[\frac{f(x, u_n^+)}{u_n^+} w_n^+ \right] w^+ \, dx$$

but $\lim_{n \to +\infty} \left[\frac{f(x, u_n^+)}{u_n^+} w_n^+ \right] = +\infty$ a.e. in Ω_2, so we must have $meas\, \Omega_2 = 0$, this implies that $\Omega = \Omega_1$, and $w^+ \equiv 0$ a.e. in Ω, it is impossible for we have proved that $w^+ \not\equiv 0$. Hence, $\{u_n\}$ is bounded in $H_0^1(\Omega)$. ∎

5. CASE OF $f(x, u) \equiv \lambda u$ AND SOME REMARKS

In this section, we show that our method also works for problem (1.1) in the special case where $f(x, u)$ is globally linear with respect to u, e.g., $f(x, u) \equiv \lambda u$, and some existence and nonexistence results are given.

We consider the following problem:

$$\begin{cases} -\Delta u = h(x) u^q + \lambda u, & \lambda > 0, \quad 0 < q < 1, \\ u \geqslant 0, & u \in H_0^1(\Omega). \end{cases} \tag{5.1}$$

For problem (5.1), we consider now the following functional, for $u \in H_0^1(\Omega)$:

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_\Omega (u^+)^2 \, dx - \frac{1}{q+1} \int_\Omega h(x)(u^+)^{q+1} \, dx. \tag{5.2}$$

It is well-known that a nonzero critical point of J is essentially a nonnegative weak solution to problem (5.1).

THEOREM 5.1. (i) *If* $h(x) \geqslant (\not\equiv) 0$, *then problem* (5.1) *has no positive solution for* $\lambda \geqslant \lambda_1$, *but for* $\lambda < \lambda_1$ *problem* (5.1) *has at least one positive solution if* $h(x) \in L^\infty(\Omega)$.

(ii) *If* $h(x) \leqslant (\not\equiv) 0$ *and* $h(x) \in L^\alpha(\Omega)$ *for some* $\alpha \in [\frac{2^*}{2^*-1-q}, +\infty]$, *furthermore, there exists* $\delta > 0$ *such that* $h(x) \leqslant -\delta$, *then problem* (5.1) *has always a nonnegative solution* $u \in H_0^1(\Omega)$ *with* $I(u) > 0$ *for all* $\lambda > \lambda_1$.

(iii) *If* $h(x) \leqslant (\not\equiv) 0$, *then problem* (5.1) *has no positive solution for* $\lambda \leqslant \lambda_1$,

(iv) *If* $h(x) \equiv 0$, *then problem* (5.1) *has positive solution only if* $\lambda = \lambda_1$.

Proof. (i) Suppose that problem (5.1) has a positive solution $u \in H_0^1(\Omega)$, then for the λ_1-eigenfunction $\varphi_1 > 0$, we have

$$\lambda_1 \int_\Omega u\varphi_1 \, dx = \int_\Omega \nabla u \cdot \nabla \varphi_1 \, dx = \int_\Omega h(x) u^q \varphi_1 \, dx + \lambda \int_\Omega u\varphi_1 \, dx$$

that is, $(\lambda_1 - \lambda) \int_\Omega u\varphi_1 \, dx = \int_\Omega h(x) u^q \varphi_1 \, dx > 0$ since $u > 0, h(x) \geqslant (\not\equiv) 0$ and $\varphi_1 > 0$, therefore, $\lambda < \lambda_1$ and so (5.1) has no positive solution if $\lambda \geqslant \lambda_1$.

On the other hand, if $\lambda < \lambda_1$ we claim that there exist $\rho > 0$, $\eta > 0$ such that

$$J(u) \geqslant \eta > 0 \qquad \text{for all } u \in H_0^1(\Omega) \text{ with } \|u\| = \rho. \tag{5.3}$$

In fact, by the definition of J and the definition of λ_1,

$$\begin{aligned} J(u) &\geqslant \frac{1}{2} \|u\|^2 - \frac{\lambda}{2\lambda_1} \|u\|^2 - \frac{|h|_\infty}{q+1} \int_\Omega |u^+|^{q+1} \, dx \\ &\geqslant \frac{1}{2} \|u\|^2 - \frac{\lambda}{2\lambda_1} \|u\|^2 - \frac{|h|_\infty C}{q+1} \|u\|^{q+1}, \end{aligned}$$

then we can find some $\rho > 0$ suitable large such that (5.3) holds. By (5.3) and the proof of Theorem 1.1, we know that (5.1) has a positive solution.

(ii) To prove this part, we apply Proposition 3.1.

Step 1. There exist $\rho > 0$, $\eta > 0$ such that $I(u)|_{\partial B_\rho} \geqslant \eta > 0$.

Indeed, noting $q+1<2<2^*$, by Hölder's and Young's inequality we have

$$\int_\Omega u^2\,dx \leqslant \left(\int_\Omega u^{q+1}\,dx\right)^{1/\alpha}\left(\int_\Omega u^{2^*}\,dx\right)^{\frac{1}{\beta}}, \qquad \text{where } \frac{1}{\alpha}+\frac{1}{\beta}=1$$
$$\leqslant \varepsilon|u|_{q+1}^{q+1}+\frac{|u|_{2^*}^{2^*}}{(\alpha\varepsilon)^{\beta/\alpha}\beta} \qquad \text{for any } \varepsilon>0.$$

Hence,

$$\begin{aligned}
J(u) &\geqslant \frac{1}{2}\,\|u\|^2-\frac{\lambda\varepsilon}{2}\,|u^+|_{q+1}^{q+1}-\frac{\lambda|u^+|_{2^*}^{2^*}}{2(\alpha\varepsilon)^{\beta/\alpha}\beta}+\frac{\delta}{q+1}\int_\Omega (u^+)^{q+1}\,dx\\
&\geqslant \frac{1}{2}\,\|u\|^2-\left(\frac{2\alpha\delta}{q+1}\right)^{-\beta/\alpha}\lambda^\beta\beta^{-1}|u^+|_{2^*}^{2^*} \qquad \text{by taking } \varepsilon=\frac{2\delta}{\lambda(q+1)}\\
&\geqslant \frac{1}{2}\,\|u\|^2-\left(\frac{2\alpha\delta}{q+1}\right)^{-\beta/\alpha}\lambda^\beta\beta^{-1}S\|u^+\|_{2^*}^{2^*} \qquad \text{by Sobolev inequality,}
\end{aligned}$$

so, this completes Step 1.

Step 2: There exists $e\notin \bar{B}_\rho$ such that $J(e)<0$.

Indeed, since $h(x)\in L^\alpha(\Omega)$, $\alpha\in[\frac{2^*}{2^*-1-q},+\infty]$ and $q\in(0,1)$, it follows that

$$\left|\int_\Omega h(x)\varphi_1(x)^{q+1}\,dx\right|<+\infty \qquad \text{and} \qquad \frac{t^{q-1}}{q+1}\int_\Omega h(x)\varphi_1^{q+1}\,dx \overset{t\to+\infty}{\to} 0,$$

where $\varphi_1>0$ be the λ_1-eigenfunction. Then, for $t>0$ we see that

$$\begin{aligned}
\lim_{t\to+\infty}\frac{I(t\varphi_1)}{t^2} &= \frac{1}{2}\,\|\varphi_1\|^2-\frac{\lambda}{2}\int_\Omega \varphi_1^2\,dx+\lim_{t\to+\infty}\frac{t^{q-1}}{q+1}\int_\Omega h(x)\varphi_1(x)^{q+1}\,dx\\
&= \frac{1}{2}\,\|\varphi_1\|^2-\frac{\lambda}{2}\int_\Omega \varphi_1^2\,dx\\
&= \frac{1}{2}\,(1-\lambda/\lambda_1)\|\varphi_1\|^2<0 \qquad \text{by } \lambda>\lambda_1,
\end{aligned}$$

and Step 2 is proved.

Step 3. The proof of Theorem 5.1(ii).

By steps 1 and 2, it follows from Proposition 3.1 that there exists $\{u_n\}\in H_0^1(\Omega)$ such that

$$\frac{1}{2}\,\|u_n\|^2-\frac{\lambda}{2}\int_\Omega (u_n^+)^2\,dx-\frac{1}{q+1}\int_\Omega h(x)(u_n^+)^{q+1}\,dx=c+o(1), \qquad (5.4)$$

$$\|u_n\|^2 - \lambda \int_\Omega (u_n^+)^2\,dx - \int_\Omega h(x)(u_n^+)^{q+1}\,dx = o(1), \tag{5.5}$$

$$\int_\Omega \nabla u_n \cdot \nabla \varphi\,dx - \lambda \int_\Omega u_n^+ \varphi\,dx + \int_\Omega h(x)(u_n^+)^q \varphi\,dx = o(1)$$
$$\text{for all } \varphi \in H_0^1(\Omega). \tag{5.6}$$

As we know, to finish our proof we need only show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$.

By contradiction, suppose that $\|u_n\| \to +\infty$ as $n \to +\infty$, and let $w_n = \frac{u_n}{\|u_n\|}$, then there exists a subsequence of $\{w_n\}$, still denoted by $\{w_n\}$, and some $w \in H_0^1(\Omega)$ such that

$$w_n \overset{n}{\rightharpoonup} w \quad \text{weakly in } H_0^1(\Omega), \qquad w_n \overset{n}{\to} w \quad \text{a.e. in } \Omega,$$

$w_n \overset{n}{\to} w$ strongly in $L^p(\Omega)$, $p \in [1, 2^*)$ if $N \geqslant 3$ and $p \in [1, \infty)$ if $N = 1, 2$. Similarly,

$$w_n^+ \overset{n}{\rightharpoonup} w^+ \quad \text{weakly in } H_0^1(\Omega), \qquad w_n^+ \overset{n}{\to} w^+ \quad \text{a.e. in } \Omega,$$

$$w_n^+ \overset{n}{\to} w^+ \qquad \text{strongly in } L^p(\Omega), \quad p \text{ is the same as above.}$$

Firstly, we claim that $w \not\equiv 0$. Otherwise, it follows from (5.5) that

$$\|w_n\|^2 - \lambda \int_\Omega (w_n^+)^2\,dx + \frac{1}{\|u_n\|^{1-q}} \int_\Omega h(x)(w_n^+)^{q+1}\,dx = o(1),$$

this implies that $\|w_n\| \to^n 0$, which contradicts that $\|w_n\| = 1$, so $w \not\equiv 0$.

Next, we claim that $w > 0$. In fact, by $\|u_n\| \overset{n}{\to} \infty$, $h(x) \in L^\alpha$, $\alpha \in [\frac{2^*}{2^*-1-q}, +\infty]$ and w_n^+ is bounded in $H_0^1(\Omega)$, it is clear that

$$\frac{1}{\|u_n\|^{1-q}} \int_\Omega h(x)(w_n^+)^q \varphi\,dx \overset{n}{\to} 0 \qquad \text{as } n \to +\infty,$$

then by (5.6) we have that

$$\int_\Omega \nabla w \cdot \nabla \varphi\,dx - \lambda \int_\Omega w^+ \varphi\,dx = 0.$$

Taking $\varphi = w^-$ in the above equality, we see that $\int_\Omega |\nabla w^-|^2\,dx = 0$, that is, $w^- \equiv 0$, and $w \equiv w^+ \geqslant 0$, then by the strong maximum principle, we know that $w > 0$ a.e. in $H_0^1(\Omega)$. Therefore,

$$\int_\Omega \nabla w \cdot \nabla \varphi \, dx = \lambda \int_\Omega w\varphi \, dx \qquad \text{for all } \varphi \in H_0^1(\Omega),$$

which is impossible since $\lambda > \lambda_1$. So, $\{u_n\}$ is bounded in $H_0^1(\Omega)$, and the proof of part (ii) is completed.

(iii) If problem (5.1) has a positive solution $u \in H_0^1(\Omega)$, then

$$\lambda_1 \int_\Omega u\varphi_1 \, dx = \int_\Omega \nabla u \cdot \nabla \varphi_1 \, dx = \lambda \int_\Omega u\varphi_1 \, dx + \int_\Omega h(x)u^q \varphi_1 \, dx,$$

where $\varphi_1 > 0$ is the λ_1-eigenfunction. Hence,

$$(\lambda_1 - \lambda) \int_\Omega u\varphi_1 \, dx = \int_\Omega h(x)u^q \varphi \, dx < 0$$

since $u > 0, \varphi_1 > 0$ and $h(x) \leqslant (\not\equiv) 0$, hence, $\lambda > \lambda_1$, and (iii) is proved.

(iv) By the properties of the λ_1-eigenfunction, we know that (iv) is true. ∎

Remark 5.1.

(1) For problem (5.1), some existence results (nonpositive solution) can be found in [21] in the case of $\lambda \leqslant \lambda_1$.

(2) By Theorem 5.1 we see that condition $(f2)$ is necessary for Theorem 1.2.

(3) A similar result to Theorem 5.1(ii) was also obtained in [29] under some weak conditions on $h(x)$, but they require another sublinear term.

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